University of Groningen
Exam Numerical Mathematics 1, July 10, 2015

Use of a simple calculator is allowed. All answers need to be motivated.
In front of the exercises you find its weight. In fact it gives the number of tenths which can be gained in the final mark. In total 5.4 points can be scored with this exam.

Exercise 1

(a) [4] Given the data pairs $(x, y) = \{(-1, 0), (-0.5, -1), (0.5, 1)\}$. Give first the general form of the interpolation polynomial expressed in the Lagrange characteristic polynomials and next indicate how it is defined for an interpolation on the given data points.

(b) [4] The conditioning of interpolation is expressed by the inequality

$$\max_{x\in I} |p_n(x) - \tilde{p}_n(x)| \leq \Lambda \max_{k\in\{0,\cdots,n\}} |y_k - \tilde{y}_k|$$

where $p_n(x)$ is the interpolation polynomial based on the pairs (x_k, y_k) and $\tilde{p}_n(x)$ on the pairs $(x_k, \tilde{y}_k)$, $k = 0, \cdots n$. Show that Lebesque's constant Λ is given by $\Lambda = \sum_{k=0}^{n} \max_{x\in I} |l_k(x)|$, where $l_k(x)$, $k = 0, \cdots n$, are the Lagrange characteristic polynomials. Give Λ for the interpolation on $[-1, 0.5]$ and data points give in part (a).

(c) [2] Define both the midpoint rule and the composite midpoint rule for an integration of a function f over an interval $[a, b]$.

(d) [4] The error of the midpoint rule is given by $E^t = -(b - a)^3 f''(\xi)/24$, for some $\xi \in [a, b]$. What is the degree of exactness of this method? Why? Show that the error of the composite midpoint rule is given by $E^c = -(b - a)H^2 f''(\zeta)/24$, for some $\zeta \in [a, b]$, where H is the length of the subintervals in $[a, b]$.
Hint: You may use that for any continuous function g it holds that there exists a ζ in [a,b] such that $ng(\zeta) = \sum_{i=1}^{n} g(x_i)$ for an arbitrary set of points x_i, $i = 1, \cdots, n$ in [a,b].

Exercise 2

(a) Consider the linear system $Ax = b$ and suppose that the matrix A is an $m \times n$ matrix with $m > n$ of full rank (i.e. the columns form an independent set of vectors) leading to an overdetermined equation.

(i) [4] One way of solving this is minimizing $(Ax - b, Ax - b)$ over x. Show that this minimization leads to $A^T Ax = A^T b$, where $A^T A$ is a square matrix of order n.

(ii) [1] What is the numerical problem with solving the equation in the previous part?

Continue on other side!

(b) Consider the iteration $x^{(k+1)} = Ax^{(k)}$, with $x^{(0)}$ given and suppose that one eigenvalue, say λ_1, of A is bigger in absolute value than all others. Moreover, A has a complete set of eigenvectors.

  (i) $\boxed{4}$ Show that $x^{(k)}$, will converge to the eigenvector associated to λ_1 if $x^{(0)}$ has a nonzero component in the direction of this eigenvector. Also indicate the convergence factor.

  (ii) $\boxed{2}$ How can we obtain an estimate of λ_1 during the iteration?

  (iii) $\boxed{3}$ Assume $|\lambda_1| \neq 1$. Depending on whether it is bigger or less than one, what will eventually happen if we perform the iteration on a computer? And what is done to prevent this situation if we are only interested in finding λ_1 and the associated eigenvector?

Exercise 3

Consider the nonlinear system $f(x) = 0$, where f is a mapping from R^n to R^n.

(a) $\boxed{4}$ Derive Newton's method for the above system and indicate which linear system has to be solved in each step.

(b) $\boxed{2}$ Suppose $f_1 = \sin(x_1 + 2x_2 - 1)$, $\quad f_2 = \arctan(x_2 - x_1)$. Give the Jacobian matrix of f.

(c) $\boxed{4}$ Zeros of functions can be found by a fixed point method $x^{(k+1)} = \phi(x^{(k)})$. Show that this fixed point method will converge if $|\phi'(\alpha)| < 1$ and $x^{(0)}$ close enough to the fixed point α.

(d) $\boxed{4}$ Derive Aitken's extrapolation formula

$$\tilde{x}^{(k+1)} = \frac{x^{(k+1)}x^{(k-1)} - (x^{(k)})^2}{x^{(k+1)} - 2x^{(k)} + x^{(k-1)}}$$

where $\tilde{x}^{(k+1)}$ is the extrapolated value based on $x^{(k-1)}$, $x^{(k)} = \phi(x^{(k-1)})$, and $x^{(k+1)} = \phi(x^{(k)}) = \phi(\phi(x^{(k-1)}))$.

Exercise 4

Consider a system of ODEs

$$\frac{d}{dt}y(t) = f(t, y(t)), \text{ with } y(0) = y_0$$

(a) Consider the method $u_{k+1} = u_{k-1} + 2\Delta t f(t_k, u_k)$.

  (i) $\boxed{4}$ State the root condition. Show that this method satisfies this condition. What does this mean for stability?

  (ii) $\boxed{4}$ Show that the local truncation error is of second order in Δt. What is the conclusion for convergence, if you combine this with part (i).

(b) $\boxed{4}$ Consider on [0, 1] for $u(x, t)$ the diffusion equation $\partial u/\partial t = \partial^2 u/\partial x^2 + x\exp(-t)$ with initial condition $u(x, 0) = \sin(\pi x)$ and boundary conditions $u(0, t) = \sin^2(t)$ and $u(1, t) = 0$. Let the grid in x-direction be given by $x_i = i\Delta x$ where $\Delta x = 1/m$. Show that, by using $\frac{\partial^2 u}{\partial x^2}(x_i, t) \approx \frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{\Delta x^2}$ in the PDE, one obtains a system of ordinary differential equations (ODEs) of the above form. Give the components of the vector function f and the initial vector.